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WAVE REGIMES ON A NONLINEARLY VISCOUS FLUID FILM FLOWING DOWN A VERTICAL PLANE

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The flow of a thin film of a nonlinearly viscous fluid whose stress tensor is modeled by a power law, flowing down a vertical plane in the field of gravity, is considered. For the case of low flow rates, an equation that describes the evolution of surface disturbances is derived in the long-wave approximation. The domain of linear stability of the trivial solution is found, and weakly nonlinear, steady-state travelling solutions of this equation are obtained. The mechanism of branching of solution families at the singular point of the neutral curve is described.

Key words: *rheological fluid, power law, downward film, evolution equation.*

1. Formulation of the Problem. Flows of thin fluid films in the field of gravity have attracted attention of researchers for more than half a century. This interest is caused, in particular, by numerous applications of such flows in various technological processes.

In the present paper, we consider a downward non-Newtonian fluid flow along a vertical plane in the field of gravity. One of the best known models of nonlinearly viscous fluids (the model of a power fluid) is used as a rheological relation. This model offers an adequate description of fluids with both pseudoplastic and dilatant properties. The main objective of the work is to derive a model equation that allows studying wave regimes of the flow of such rheological films.

We consider a downward flow of a thin film of a nonlinearly viscous fluid over a vertical plane in the field of gravity. The schematic of the flow and the coordinate system used are shown in Fig. 1.

The system of the Navier–Stokes and continuity equations that describe the motion of such a film has the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\nabla)\boldsymbol{u} = -\frac{\nabla p}{\rho} + \boldsymbol{g} + \frac{1}{\rho}\,\mathrm{Div}\,\tau, \qquad \mathrm{div}\,\boldsymbol{u} = 0, \tag{1}$$

where g is the acceleration of gravity, ρ is the density of the fluid, and $\mathrm{Div}\,\tau$ is the divergence of the stress tensor. Rheology of the fluid is modeled by a power law whose invariant form is [1]

$$\tau_{ik} = 2\mu_n (2D_{kl}D_{kl})^{(n-1)/2} D_{ik}, \qquad D_{ik} = \frac{1}{2}\Big[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big].$$

Here τ_{ik} is the stress tensor and D_{ik} is the strain-rate tensor. The constant μ_n is an indicator of the fluid consistency, and the parameter n characterizes the degree of non-Newtonian behavior.

The greater the deviation of n from unity, the more pronounced the viscosity anomaly in such a medium [2]. The values $0 < n < 1$ refer to pseudoplastic fluids whose apparent viscosity decreases with increasing shear rates, and the values $n > 1$ correspond to dilatant fluids whose apparent viscosity increases with increasing shear rates.

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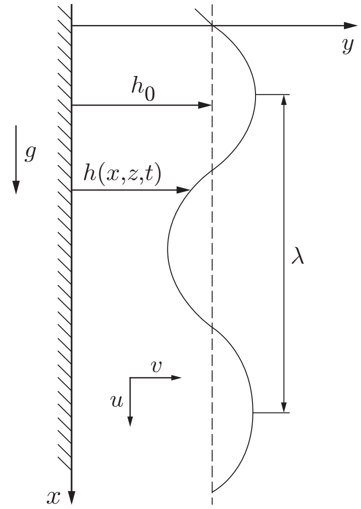

Fig. 1. Diagram of the flow.

For arbitrary flow rates of the fluid, problem (1) admits solutions with a flat free surface $h(x, z, t) = h_0$, which satisfy the no-slip conditions on the solid wall and the absence of shear stresses on the free boundary. In this case, the velocity profile is

$$U(y) = U_0\Big[1 - \Big(1 - \frac{y}{h_0}\Big)^{(n+1)/n}\Big], \qquad U_0 = \frac{n}{n+1}\Big(\frac{\rho g}{\mu_n}\Big)^{1/n} h_0^{(n+1)/n}. \tag{2}$$

Even for low flow rates, however, flow (2) can become wavy because of instability. To describe such flow regimes, we write the equations of motion in the dimensionless form. Let L be the characteristic longitudinal scale of the disturbance. Then, using L/U_0, U_0, and $\rho g h_0$ as scales for time, velocity, and pressure, and L and h_0 as scales in the x, z, and y directions, respectively, we write the equations of motion for the film in the following form (the signs of dimensionless quantities are omitted):

$$\begin{gathered}
\varepsilon\,\frac{\partial u}{\partial t} + \varepsilon u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + \varepsilon w\,\frac{\partial u}{\partial z} = \frac{1}{\mathrm{Fr}}\Big(1 - \varepsilon\,\frac{\partial p}{\partial x}\Big) + \frac{1}{\mathrm{Re}}\Big(\varepsilon\,\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \varepsilon\frac{\partial \tau_{xz}}{\partial z}\Big), \\
\varepsilon\,\frac{\partial v}{\partial t} + \varepsilon u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + \varepsilon w\,\frac{\partial v}{\partial z} = -\frac{1}{\mathrm{Fr}}\,\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\,\Big(\varepsilon\,\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \varepsilon\,\frac{\partial \tau_{yz}}{\partial z}\Big), \\
\varepsilon\,\frac{\partial w}{\partial t} + \varepsilon u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + \varepsilon w\,\frac{\partial w}{\partial z} = -\frac{\varepsilon}{\mathrm{Fr}}\,\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\Big(\varepsilon\,\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \varepsilon\,\frac{\partial \tau_{zz}}{\partial z}\Big), \\
\varepsilon\,\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \varepsilon\,\frac{\partial w}{\partial z} = 0.
\end{gathered} \tag{3}$$

The dynamic boundary conditions on the solid boundary ($y = 0$) and free boundary [$y = h(x, z, t)$] have the form

$$\begin{gathered}
u = v = w = 0, \qquad y = 0, \\
(p - p_0 - \mathrm{We}\,(K_1 + K_2))n_i - (\mathrm{Fr}\,/\,\mathrm{Re})\tau_{ik} n_k = 0, \qquad y = h(x, z, t).
\end{gathered} \tag{4}$$

Here u, v, and w are the velocity components in the x, y, and z directions, respectively, p is the pressure in the fluid, p_0 is the external pressure (without loss of generality, it can be assumed to equal zero in what follows), n_i are the components of the normal vector

$$\boldsymbol{n} = (-\varepsilon h_x, 1, -\varepsilon h_z)/\sqrt{1 + \varepsilon^2 h_x^2 + \varepsilon^2 h_z^2},$$

K_i are the dimensionless main curvatures of the free surface

$$K_1 + K_2 = \frac{(1+\varepsilon^2 h_x^2)\varepsilon h_{zz} - 2\varepsilon^3 h_x h_z h_{xz} + (1+\varepsilon^2 h_z^2)\varepsilon h_{xx}}{(1+\varepsilon^2 h_x^2 + \varepsilon^2 h_z^2)^{3/2}}$$

(hereinafter, the subscripts at the variable h indicate differentiation with respect to the corresponding variable), and τ_{ik} are the components of the stress tensor

$$\tau_{xx} = 2I^{(n-1)/2}\varepsilon\,\frac{\partial u}{\partial x}, \qquad \tau_{yy} = 2I^{(n-1)/2}\,\frac{\partial v}{\partial y}, \qquad \tau_{zz} = 2I^{(n-1)/2}\varepsilon\,\frac{\partial w}{\partial z},$$

$$\tau_{xy} = \tau_{yx} = I^{(n-1)/2}\Big(\frac{\partial u}{\partial y} + \varepsilon\,\frac{\partial v}{\partial x}\Big), \qquad \tau_{xz} = \tau_{zx} = I^{(n-1)/2}\varepsilon\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big), \tag{5}$$

$$\tau_{yz} = \tau_{zy} = I^{(n-1)/2}\Big(\varepsilon\,\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\Big),$$

where I is the second invariant of the strain-rate tensor

$$I = 2\Big[\varepsilon^2\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2 + \varepsilon^2\Big(\frac{\partial w}{\partial z}\Big)^2\Big] + \Big(\frac{\partial u}{\partial y} + \varepsilon\,\frac{\partial v}{\partial x}\Big)^2 + \varepsilon^2\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big)^2 + \Big(\varepsilon\,\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\Big)^2.$$

The following quantities are used as parameters in (3)–(5): $\varepsilon = h_0/L$, Reynolds number $\mathrm{Re} = \rho\, h_0^n/(\mu_n U_0^{n-2})$, Froude number $\mathrm{Fr} = U_0^2/(g\, h_0)$, and Weber number $\mathrm{We} = \sigma/(\rho g h_0^2)$.

The free boundary also obeys the kinematic condition

$$\varepsilon\Big(\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} + w\,\frac{\partial h}{\partial z}\Big) = v, \qquad y = h(x,z,t). \tag{6}$$

Using Eq. (2), we can easily show that the following relation is satisfied with the above-made choice of characteristic scales:

$$\mathrm{Re} = ((n+1)/n)^n\,\mathrm{Fr}\,.$$

In what follows, we confine ourselves to long-wave disturbances, assuming that $\varepsilon \ll 1$ and Reynolds numbers are rather low: $\mathrm{Re} \approx 1$.

To apply the multiple scale method (see [3]), we introduce a set of fast and slow times and new functions:

$$\begin{gathered}\tau_m = \varepsilon^m t, \qquad m = 0,1,2,\dots\ , \\ u = U + \varepsilon u', \qquad v = \varepsilon v', \qquad w = \varepsilon^2 w', \qquad p = p_0 + \varepsilon p', \qquad h = 1 + \varepsilon h'.\end{gathered} \tag{7}$$

Neglecting terms of the order of ε^2 and higher and transposing the boundary conditions from the free boundary to its undisturbed level (i.e., expanding all functions in powers of $\varepsilon h'$), we obtain the system (the primes at disturbed quantities are omitted)

$$\begin{gathered}\varepsilon\Big(\frac{\partial u}{\partial \tau_0} + U\,\frac{\partial u}{\partial x} + v\,\frac{dU}{dy} + \frac{1}{\mathrm{Fr}}\,\frac{\partial p}{\partial x}\Big) \\ = \frac{n}{\mathrm{Re}}\,\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y}\Big(\frac{dU}{dy}\Big)^{n-1}\Big) + (n-1)\,\frac{\varepsilon}{\mathrm{Re}}\,\frac{\partial}{\partial y}\Big(\Big[n\Big(\frac{\partial u}{\partial y}\Big)^2 + \frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big]\Big(\frac{dU}{dy}\Big)^{n-2}\Big), \\ \frac{1}{\mathrm{Fr}}\,\frac{\partial p}{\partial y} = \frac{\varepsilon}{\mathrm{Re}}\Big(\Big[n\,\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial w}{\partial y}\Big)\Big]\Big(\frac{dU}{dy}\Big)^{n-1} + 2\,\frac{\partial}{\partial y}\Big(\frac{\partial v}{\partial y}\Big(\frac{dU}{dy}\Big)^{n-1}\Big)\Big), \\ \varepsilon\Big(\frac{\partial w}{\partial \tau_0} + U\,\frac{\partial w}{\partial x} + \frac{1}{\mathrm{Fr}}\,\frac{\partial p}{\partial z}\Big) = \frac{1}{\mathrm{Re}}\,\frac{\partial}{\partial y}\Big(\frac{\partial w}{\partial y}\Big(\frac{dU}{dy}\Big)^{n-1}\Big) + (n-1)\,\frac{\varepsilon}{\mathrm{Re}}\,\frac{\partial}{\partial y}\Big(\Big(\frac{\partial u}{\partial y}\,\frac{\partial w}{\partial y}\,\frac{dU}{dy}\Big)^{n-2}\Big); \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\end{gathered} \tag{8}$$

with the following boundary conditions:

$$u = v = w = 0, \qquad y = 0,$$

$$0 = \Big(nu_y\Big(\frac{dU}{dy}\Big)^{n-1} - \Big(\frac{n+1}{n}\Big)^n h\Big) + \varepsilon\Big(nh\,\frac{\partial}{\partial y}\Big[u_y\Big(\frac{dU}{dy}\Big)^{n-1}\Big]$$
$$+ (n-1)\Big(\frac{dU}{dy}\Big)^{n-2}\Big[n\Big(\frac{\partial u}{\partial y}\Big)^2 + \frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big]\Big), \qquad y = 1,$$
$$\varepsilon\Big(p + \varepsilon\,\frac{\partial p}{\partial y}\,h - \mathrm{We}\,\varepsilon^2\Delta h\Big) = 2\,\frac{\mathrm{Fr}}{\mathrm{Re}}\,\varepsilon^2\Big(\frac{dU}{dy}\Big)^{n-1}\frac{\partial v}{\partial y}, \qquad y = 1, \tag{9}$$
$$0 = \Big(w_y\Big(\frac{dU}{dy}\Big)^{n-1}\Big) + \varepsilon\Big(h\,\frac{\partial}{\partial y}\Big[w_y\Big(\frac{dU}{dy}\Big)^{n-1}\Big] + (n-1)\Big(\frac{dU}{dy}\Big)^{n-2}\frac{\partial u}{\partial y}\,\frac{\partial w}{\partial y}\Big), \qquad y = 1.$$

In condition (9), $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$ is the Laplace operator. Terms of higher orders of ε are retained here because the values of We are normally high for thin films of numerous fluids; hence, we assume that $\mathrm{We} \gg 1$ and $\mathrm{We}\,\varepsilon^2 \approx 1$.

The kinematic condition (6) now acquires the form

$$v + \varepsilon\,\frac{\partial v}{\partial y}\,h = h_{\tau_0} + \varepsilon h_{\tau_1} + h_x + \varepsilon u h_x + \varepsilon w h_z, \qquad y = 1. \tag{10}$$

We seek for the solution of problem (8), (9) in the form of series with respect to ε:

$$(u, v, p, h) = \sum_{m=0}^{\infty} \varepsilon^m (u^m, v^m, p^m, h^m). \tag{11}$$

Equating the coefficients at identical powers of ε in the initial system to zero, we obtain the following expressions for the zeroth order:

$$u^0(x,y,z,t) = \frac{n+1}{n}\,(1 - (1-y)^{1/n})h^0,$$
$$v^0(x,y,z,t) = -\frac{n+1}{n}\Big[y + \frac{n}{n+1}\,((1-y)^{(n+1)/n} - 1)\Big]h_x^0, \tag{12}$$
$$w^0(x,y,z,t) = 0, \qquad p^0\Big|_{y=1} = \mathrm{We}\,\varepsilon^2\Delta h^0.$$

Substituting Eq. (12) into Eq. (10), we obtain the equation that describes the behavior of disturbances in the first approximation, namely,

$$h_{\tau_0} + c_0 h_x = 0, \qquad c_0 = (n+1)/n.$$

It follows from here that all disturbances in this approximation propagate with a constant velocity, which exceeds the flow velocity at the flat free boundary by a factor of c_0, i.e.,

$$h = h(\xi), \qquad \xi = x - c_0\tau_0.$$

Considering the next approximation, after cumbersome but simple calculations, we obtain the expressions for the next terms of series (11):

$$u^1(x,y,z,t) = c_0^{1-n}\Big(c_0^n h^1[1 - (1-y)^{1/n}]$$
$$- \frac{\mathrm{Re}}{n}\Big[(1-yz)^{(n+1)/n} - \frac{n}{2(2n+1)}(1-z)^{(2n+2)/n} - \frac{3n+2}{2(2n+2)}\Big]h_x^0$$
$$+ \frac{\mathrm{Re}}{\mathrm{Fr}}\,\frac{\partial p^0}{\partial x}\,\frac{1}{n+1}\,[(1-z)^{(n+1)/n} - 1] + c_0^n\,\frac{(h^0)^2}{n}\,[1 - (1-z)^{(1-n)/n}]\Big),$$
$$v^1(x,y,z,t) = -c_0^{1-n}\Big(c_0^n I_1 h_x^1 - \frac{\mathrm{Re}}{n}\,h_{xx}^0 I_2 + \frac{\mathrm{Re}}{\mathrm{Fr}}\,\frac{I_3}{n+1}\,\frac{\partial^2 p^0}{\partial x^2} + 2c_0^n\,\frac{I_4}{n} + \frac{n}{n+1}\,\frac{\mathrm{Re}}{\mathrm{Fr}}\,\frac{\partial^2 p^0}{\partial y^2}\,I_3\Big),$$
$$w^1(x,y,z,t) = c_0^{1-n}\,\frac{n}{n+1}\,\frac{\mathrm{Re}}{\mathrm{Fr}}\,\frac{\partial p^0}{\partial y}\,[(1-y)^{(n+1)/n} - 1]. \tag{13}$$

Here

$$I_1 = y + \frac{n}{n+1}\,[(1-y)^{(n+1)/n} - 1],$$

$$I_2 = \frac{n}{2n+1}\,[1-(1-y)^{(2n+1)/n}] - \frac{n^2}{2(2n+1)(3n+2)}\,[1-(1-y)^{(3n+2)/n}] - \frac{3n+2}{2(2n+1)}\,y,$$

$$I_3 = \frac{n}{2n+1}\,[1-(1-y)^{(2n+1)/n}] - y, \qquad I_4 = y + n[(1-y)^{1/n} - 1].$$

Substituting Eq. (13) into formula (10), we obtain the nonlinear equation for determining h:

$$h_{\tau_1} + 2\,\frac{c_0}{n}\,hh_x + \frac{2(n+1)^2c_0^{1-n}}{n(2n+1)(3n+2)}\,\mathrm{Re}\,h_{xx} + \frac{c_0^{1-n}}{2n+1}\,\frac{\mathrm{Re}}{\mathrm{Fr}}\,\mathrm{We}\,\varepsilon^2(\Delta h_{xx} + n\Delta h_{zz}) = 0. \tag{14}$$

This equation describes the evolution of spatial disturbances on a vertical film of a power-type fluid in a reference system moving with a velocity c_0 relative to the wall.

Using the substitutions

$$h = aH, \qquad x_1 = bx, \qquad z_1 = bz, \qquad \tau = dt, \tag{15}$$

where

$$a = \frac{4(n+1)^3c_0^{-n}}{(2n+1)(3n+2)}\,\mathrm{Re}\sqrt{\frac{2\,\mathrm{Fr}}{n(3n+2)\,\mathrm{We}\,\varepsilon^2}},$$

$$b = (n+1)\sqrt{\frac{2\,\mathrm{Fr}}{n(3n+2)\,\mathrm{We}\,\varepsilon^2}}, \qquad d = \frac{4(n+1)^4c_0^{1-n}}{n^2(2n+1)(3n+2)^2}\,\frac{\mathrm{Re}\,\mathrm{Fr}}{\mathrm{We}\,\varepsilon^2},$$

we transform Eq. (14) to

$$\frac{\partial H}{\partial \tau} + 4H\,\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2} + \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\Big)\Big(\frac{\partial^2}{\partial x^2} + n\,\frac{\partial^2}{\partial z^2}\Big)H = 0. \tag{16}$$

Transformations (15) mean, in particular, that the small parameter ε used in the expansion is

$$\varepsilon = \sqrt{\frac{2(n+1)^2}{n(3n+2)c_0^n}\,\frac{\mathrm{Re}}{\mathrm{We}}} \tag{17}$$

and, correspondingly, the characteristic longitudinal size of disturbances is determined by the equality

$$L = \sqrt{\frac{n(3n+2)c_0^n}{2(n+1)^2}\,\frac{\mathrm{We}}{\mathrm{Re}}}\,h_0.$$

As is seen from Eq. (17), the long-wave assumption for disturbances considered is valid for high values of the Weber number, as in the case of Newtonian fluids.

In the case of two-dimensional disturbances, Eq. (16) coincides with the equation that describes the waves on the surface of a Newtonian fluid film [4]:

$$\frac{\partial H}{\partial \tau} + 4H\,\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2} + \frac{\partial^4 H}{\partial x^4} = 0. \tag{18}$$

This equation is widely known as the Kuramoto–Sivashinsky (KS) equation. The KS equation has been studied in much detail, and many of its solutions are known (see, e.g., [5, 6]). For $n = 1$, Eq. (16) transforms to the equation from [7], which describes spatial waves for the Newtonian fluid.

Equation (16) is a typical example of model equations that arise in studying the evolution of disturbances in actively dissipative media. Indeed, the linear analysis of stability of the trivial solution $H = 0$ shows that it is unstable to disturbances of the form $\exp\,[i\alpha(x - c\tau) + i\beta z]$ with components of the wave vector (α, β) lying inside the domain bounded by the neutral curve

$$n\beta^4 + (n+1)\alpha^2\beta^2 + \alpha^4 - \alpha^2 = 0. \tag{19}$$

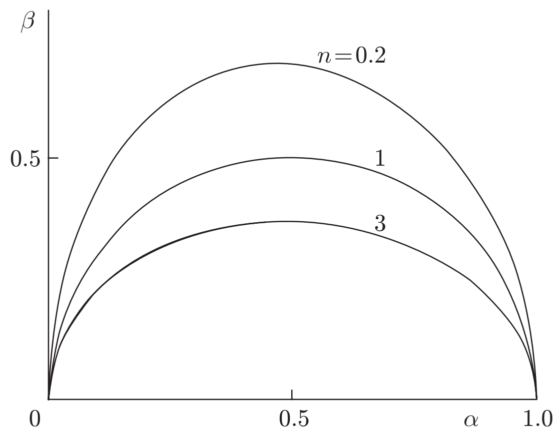

Fig. 2. Neutral curves for different values of the parameter n.

Such disturbances increase exponentially [whereas disturbances with wavenumbers outside the domain bounded by curve (19) decay]. Owing to nonlinear effects, further growth of unstable disturbances is terminated, which can result in formation of steady-state travelling modes. Curves (19) for several values of n are shown in Fig. 2.

2. Analytical Results. Obviously, Eq. (16) has solutions propagating at an angle to the free-stream direction (to the x axis). Indeed, using the substitutions

$$\xi_1 = a_1(x + b_1 z), \qquad t_1 = a_1^2\tau, \qquad a_1^4 = 1/((1+b_1^2)(1+nb_1^2)), \qquad H_1 = a_1 H(t_1, \xi_1),$$

we transform Eq. (16) to Eq. (18). It follows from here that Eq. (16) has solutions in the form of two-dimensional waves propagating at an angle to the x axis, the tangent of this angle satisfying the relation $\tan\psi = b_1$. Thus, all these solutions, which are actually one-dimensional, are obtained by simple recalculation of the solutions of the KS equation; therefore, in what follows, we confine ourselves to studying the solutions of Eq. (16) travelling in the streamwise direction.

For steady-state travelling solutions $H = H(\xi, z)$, $\xi = x - ct$, Eq. (16) becomes

$$-c\,\frac{\partial H}{\partial \xi} + 4H\,\frac{\partial H}{\partial \xi} + \frac{\partial^2 H}{\partial \xi^2} + \Big(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial z^2}\Big)\Big(\frac{\partial^2}{\partial \xi^2} + n\,\frac{\partial^2}{\partial z^2}\Big)H = 0. \tag{20}$$

We consider periodic solutions of Eq. (20) with the wavenumbers α and β in the ξ and z directions, respectively.

As Eq. (20) is invariant with respect to the transformations

$$H \to H + \text{const}, \qquad c \to c + 4\text{const},$$

$$H \to -H, \qquad c \to -c, \qquad \xi \to -\xi,$$

we consider only solutions symmetric in terms of z

$$H(\xi, z) = H(\xi, -z), \tag{21}$$

for which the following normalization condition is valid:

$$c \geq 0, \qquad \int\limits_0^{\lambda_z}\int\limits_0^{\lambda_\xi} H(\xi, z)\,d\xi\,dz = 0, \qquad \lambda_\xi = \frac{2\pi}{\alpha}, \qquad \lambda_z = \frac{2\pi}{\beta}. \tag{22}$$

In the plane (α, β), spatial periodic solutions of Eq. (20) with an infinitely small amplitude branch off from the trivial solutions along curve (19). We can logically assume that the solutions of small but finite amplitude exist in the vicinity of this curve. Therefore, for wavenumbers in this vicinity, we seek the solution in the form of a series in terms of the small parameter ϵ

$$H = \epsilon H_1 + \epsilon^2 H_2 + \ldots\ , \qquad c = \epsilon c_1 + \epsilon^2 c_2 + \ldots\ . \tag{23}$$

Following [3], we introduce the set of fast and slow variables

$$\xi_m = \epsilon^m \xi, \qquad z_m = \epsilon^m z, \qquad m = 0, 1, 2, \dots \ .$$

Then, the differentiation operations in (20) are presented as

$$\begin{gathered}
\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi_0} + \epsilon \frac{\partial}{\partial \xi_1} + \epsilon^2 \frac{\partial}{\partial \xi_2} + \dots \ , \\
\frac{\partial^2}{\partial \xi^2} = \frac{\partial^2}{\partial \xi_0^2} + 2\epsilon \frac{\partial^2}{\partial \xi_0\, \partial \xi_1} + \epsilon^2 \Big(\frac{\partial^2}{\partial \xi_1^2} + 2 \frac{\partial^2}{\partial \xi_0\, \partial \xi_2} \Big) + \dots \ , \\
\frac{\partial}{\partial z} = \frac{\partial}{\partial z_0} + \epsilon \frac{\partial}{\partial z_1} + \epsilon^2 \frac{\partial}{\partial z_2} + \dots \ , \\
\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z_0^2} + 2\epsilon \frac{\partial^2}{\partial z_0\, \partial z_1} + \epsilon^2 \Big(\frac{\partial^2}{\partial z_1^2} + 2 \frac{\partial^2}{\partial z_0\, \partial z_2} \Big) + \dots \ , \\
\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \Big) \Big(\frac{\partial^2}{\partial x^2} + n \frac{\partial^2}{\partial z^2} \Big) = \Big[\frac{\partial^4}{\partial \xi_0^4} + (n+1) \frac{\partial^4}{\partial \xi_0^2\, \partial z_0^2} + n \frac{\partial^4}{\partial z_0^4} \Big] \\
+ \epsilon \Big[4 \frac{\partial^4}{\partial \xi_0^3\, \partial \xi_1} + 4n \frac{\partial^4}{\partial z_0^3\, \partial z_1} + (2n+2) \frac{\partial^4}{\partial \xi_0^2\, \partial z_0\, \partial z_1} + (2n+2) \frac{\partial^4}{\partial z_0^2\, \partial \xi_0\, \partial \xi_1} \Big] \\
+ \epsilon^2 \Big[4 \Big(\frac{\partial^2}{\partial \xi_0\, \partial \xi_1} + n \frac{\partial^2}{\partial z_0\, \partial z_1} \Big) \Big(\frac{\partial^2}{\partial \xi_0\, \partial \xi_1} + \frac{\partial^2}{\partial z_0\, \partial z_1} \Big) \\
+ \Big(\frac{\partial^2}{\partial \xi_0^2} + n \frac{\partial^2}{\partial z_0^2} \Big) \Big(\frac{\partial^2}{\partial \xi_1^2} + 2 \frac{\partial^2}{\partial \xi_0\, \partial \xi_2} + \frac{\partial^2}{\partial z_1^2} + 2 \frac{\partial^2}{\partial z_0\, \partial z_2} \Big) \\
+ \Big(\frac{\partial^2}{\partial \xi_0^2} + \frac{\partial^2}{\partial z_0^2} \Big) \Big(\frac{\partial^2}{\partial \xi_1^2} + 2 \frac{\partial^2}{\partial \xi_0\, \partial \xi_2} + n \frac{\partial^2}{\partial z_1^2} + 2n \frac{\partial^2}{\partial z_0\, \partial z_2} \Big) \Big] + \dots \ .
\end{gathered} \tag{24}$$

Substituting series (23) into (20) and collecting [with allowance for Eq. (24)] terms at identical powers of ϵ, we obtain an infinite system of linear differential equations. The first order in this system corresponds to the equation

$$\frac{\partial^2 H_1}{\partial \xi_0^2} + \Big(\frac{\partial^2}{\partial \xi_0^2} + \frac{\partial^2}{\partial z_0^2} \Big) \Big(\frac{\partial^2}{\partial \xi_0^2} + n \frac{\partial^2}{\partial z_0^2} \Big) H_1 = 0.$$

We require that the solution satisfies conditions (22) and obtain

$$H_1 = (A\, \mathrm{e}^{i\alpha\xi_0} + \bar{A}\, \mathrm{e}^{-i\alpha\xi_0})(a\, \mathrm{e}^{i\beta z_0} + \bar{a}\, \mathrm{e}^{-i\beta z_0}), \tag{25}$$

where A and a are functions of slow coordinates; the bar indicates complex conjugation; α and β belong to the neutral curve (19). For the sought solution to satisfy the requirement of symmetry (21), A should be a function of $\xi_1, \xi_2, \dots$ only, and a should depend only on $z_1, z_2, \dots$.

The second order is described by the equation

$$\begin{gathered}
-c_1 \frac{\partial H_1}{\partial \xi_0} + 2 \frac{\partial H_1^2}{\partial \xi_0} + 2 \frac{\partial^2 H_1}{\partial \xi_0\, \partial \xi_1} + 4 \Big[\frac{\partial^4 H_1}{\partial^3 \xi_0 \partial \xi_1} + n \frac{\partial^4 H_1}{\partial^3 z_0\, \partial z_1} + \frac{n+1}{2} \frac{\partial^4 H_1}{\partial^2 \xi_0\, \partial z_0\, \partial z_1} \\
+ \frac{n+1}{2} \frac{\partial^4 H_1}{\partial^2 z_0\, \partial \xi_0\, \partial \xi_1} \Big] + \frac{\partial^2 H_2}{\partial \xi_0^2} + \Big(\frac{\partial^2}{\partial \xi_0^2} + \frac{\partial^2}{\partial z_0^2} \Big) \Big(\frac{\partial^2}{\partial \xi_0^2} + n \frac{\partial^2}{\partial z_0^2} \Big) H_2 = 0.
\end{gathered} \tag{26}$$

Here, we should make the following comment. Determination of the form of A and a by considering higher approximations allows us to find the relation between the corrections to the wave-vector components α and β and the absolute value of the amplitude of the first harmonic. Knowing, e.g., the correction $\Delta\alpha$ to α, we can say that we found a solution in the vicinity of another point on the neutral curve (19) for which $\alpha_1 = \alpha + \Delta\alpha$. Therefore, we can assume that $A = \text{const}$ in Eq. (25) and include it into a. In other words, we seek for finite-amplitude solutions

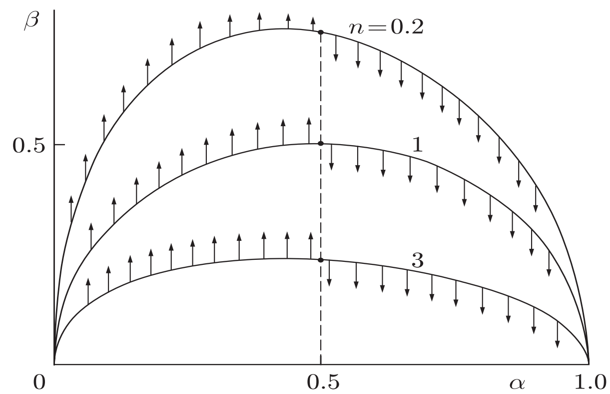

Fig. 3. Character of branching.

with a rigorously fixed value of the α component of the wave vector. This can be done almost everywhere except for vicinities of some particular points. With allowance for this comment, the condition of the absence of secular terms in Eq. (26) yields the relations

$$\frac{\partial a}{\partial z_1} = 0, \qquad c_1 = 0,$$

and the solution of Eq. (26) that satisfies the normalization conditions (22) has the form

$$H_2 = -\frac{i}{3\alpha}\, \mathrm{e}^{i2\alpha\xi_0}(a^2\,\mathrm{e}^{i2\beta z_0} + \bar{a}^2\,\mathrm{e}^{-i2\beta z_0}) - \frac{2|a|^2 i}{\alpha(4\alpha^2-1)}\, \mathrm{e}^{i2\alpha\xi_0} + \mathrm{c.c} \tag{27}$$

(hereinafter, "c.c." indicates complex conjugation).

The third-order approximation in terms of ϵ yields the equation

$$-c_2\,\frac{\partial H_1}{\partial \xi_0} + 4\,\frac{\partial H_1 H_2}{\partial \xi_0} + 4\Big[n\,\frac{\partial^4 H_1}{\partial^3 z_0\,\partial z_2} + \frac{n+1}{2}\,\frac{\partial^4 H_1}{\partial^2\xi_0\,\partial z_0\,\partial z_2}\Big] + \frac{\partial^2 H_3}{\partial \xi_0^2} + \Big(\frac{\partial^2}{\partial \xi_0^2} + \frac{\partial^2}{\partial z_0^2}\Big)\Big(\frac{\partial^2}{\partial \xi_0^2} + n\,\frac{\partial^2}{\partial z_0^2}\Big) H_3 = 0. \tag{28}$$

Identifying secular terms in Eq. (28) and equating them to zero, we obtain

$$c_2 = 0, \qquad i\beta(2n\beta^2 + (n+1)\alpha^2)\,\frac{\partial a}{\partial z_2} - \frac{2(4\alpha^2+5)}{3(4\alpha^2-1)}\, a|a|^2 = 0. \tag{29}$$

The solution of Eq. (29) has the form $a = a_0\,\mathrm{e}^{i\phi z_2}$.

It follows from two last expressions that the correction ϕ to the wave-vector component β is related to the amplitude a_0 as follows:

$$\phi = -\frac{2(4\alpha^2+5)|a_0|^2}{3\alpha^2\beta(4\alpha^2-1)\sqrt{(n-1)^2\alpha^2+4n}}. \tag{30}$$

As it follows from (25), (27), and (30), we can use the absolute value of the amplitude α_0 as the small parameter ϵ. It is seen from Eq. (30) that the correction ϕ is negative for all values of n and $\alpha > 0.5$. This means that solutions of small but finite amplitude branch off from the trivial solution to the domain of its linear instability: soft type of branching. For $\alpha < 0.5$, the correction ϕ is positive, i.e., the branching proceeds toward the domain of linear stability of the trivial solution: stiff type of branching. The situation is illustrated in Fig. 3.

As β tends to zero and α tends to unity, solution (27) remains bounded, but the value for the correction ϕ increases unlimitedly. In this domain, the solution can be constructed by fixing the values of the β component of the wave vector and seeking for corrections to α. In this situation, we can also construct a uniformly suitable expansion (23).

As α tends to 0.5, the value of β tends to

$$\beta^* = \sqrt{-\frac{n+1}{8n} + \frac{1}{8n}\sqrt{(n-1)^2 + 16n}}\,.$$

In addition to the unlimited increase in the correction ϕ, solution (27) also increases unlimitedly, and expansion (25), (27), (29) becomes invalid. The reason is that, for the solution with the wave vector of the first harmonic in the vicinity $(\alpha = 0.5, \beta^*)$, the wave vector of one harmonic of the second approximation ($2\alpha = 1$ and $\beta = 0$) also lies in the vicinity of the neutral curve (19). This means that the vicinity of the point $(\alpha = 0.5, \beta^*)$ has to be considered separately.

In the vicinity of this point, we should use the following expression instead (25):

$$H_1 = A\,\mathrm{e}^{i\xi_0/2}(a\,\mathrm{e}^{i\beta^* z_0} + \bar{a}\,\mathrm{e}^{-i\beta^* z_0}) + N\,\mathrm{e}^{i\xi_0} + \text{c.c.} \tag{31}$$

As we choose a particular point on the neutral curve (19), we cannot assume that $A = \text{const}$ in Eq. (31): it has to be considered as a function of slow coordinates ξ_m, $m = 1, 2, \ldots$.

From the second-order approximation for H_2, we obtain the expression

$$H_2 = A_1\,\mathrm{e}^{i\alpha\xi_0}(a^2\,\mathrm{e}^{i2\beta^* z_0} + \bar{a}^2\,\mathrm{e}^{-i2\beta^* z_0}) + A_2\,\mathrm{e}^{i(3/2)\alpha\xi_0}(a\,\mathrm{e}^{i\beta^* z_0} + \bar{a}\,\mathrm{e}^{-i\beta^* z_0}) + N_1\,\mathrm{e}^{i2\alpha\xi_0} + \text{c.c.}, \tag{32}$$

where

$$A_1 = -\frac{iA^2}{2\beta^{*2}((n+1) + 4n\beta^{*2})}, \quad A_2 = -\frac{6iAN}{45/16 + \beta^{*2}((9/4)(n+1) + n\beta^{*2})}, \quad N_1 = -\frac{i}{3}\,N^2,$$

and the requirement of the absence of secular terms yields the system

$$\begin{gathered} a\,\frac{\partial N}{\partial \xi_1} + c_1 N - 4A^2|a|^2 = 0, \\ \beta^*\Big(4n\beta^{*2} + \frac{n+1}{2}\Big)A\,\frac{\partial a}{\partial z_1} - \Big(\frac{1}{2} - (n+1)\beta^{*2}\Big)a\,\frac{\partial A}{\partial \xi_1} + \frac{c_1}{2}\,Aa - 2\bar{A}Na = 0. \end{gathered} \tag{33}$$

Solutions of system (33) for which (31) and (32) satisfy the conditions of symmetry (21) and normalization (22) have the form

$$A = A_0\,\mathrm{e}^{i\delta\xi_1}, \qquad a = a_0\,\mathrm{e}^{i\phi z_1}, \qquad N = N_0\,\mathrm{e}^{2i\delta\xi_1}. \tag{34}$$

Here, the amplitudes A_0, a_0, and N_0 are functions of coordinates slower than ξ_1 and z_1. Substituting Eq. (34) into Eq. (33), we obtain the system relating the corrections to the wavenumbers δ and ϕ with the values of the amplitudes A_0, a_0, and N_0

$$\begin{gathered} -(c_1 + 4i\delta)N_0 + 4A_0^2|a_0|^2 = 0, \\ A_0 a_0(-c_1 + 4(\bar{A}_0/A_0)N_0 - f(n)i\phi + g(n)i\delta) = 0, \end{gathered} \tag{35}$$

where

$$f(n) = 2\beta^{*2}(4n\beta^{*2} + (n+1)/2) > 0, \qquad g(n) = 1 - 2(n+1)\beta^{*2} \leq 0.$$

The condition of existence of nontrivial solutions of system (33), in particular, yields the relation

$$-c_1^2 + [(g-4)\delta - f\phi]ic_1 + 16|A_0|^2|a_0|^2 + 4f\phi\delta - 4g\delta^2 = 0. \tag{36}$$

In contrast to the above-considered solution (25), (27), (30), Eq. (36) is satisfied in two essentially different cases. In the first case, where $c_1 = 0$, we have

$$\phi = (g(n)\delta - 4|A_0 a_0|^2/\delta)/f(n), \qquad N_0 = -iA_0^2|a_0|^2/\delta. \tag{37}$$

The geometry of the domain of existence of solutions of this type in the plane (δ, ϕ) for $n \neq 1$ is shown in Fig. 4 as the hatched region. It follows from Eq. (37) that this solution branches off from the trivial solution along the line $\phi = g(n)\delta/f(n)$, which is a tangent to the neutral curve at the singular point $(\alpha = 0.5, \beta^*)$. Comparing Eqs. (25), (27) and (31), (32), (37), we can easily see that these solutions belong to one family. By analogy with the classification of the wave modes of the Newtonian fluid film flow (see, e.g., [8, 9]), we will call it the first spatial family.

Of interest is the second case where $c_1 \neq 0$, which requires that

$$\phi = \frac{g(n) - 4}{f(n)}\,\delta. \tag{38}$$

With allowance for Eq. (38), we obtain the following expression from Eq. (35):

$$c_1^2 = 16(|A_0|^2|a_0|^2 - \delta^2). \tag{39}$$

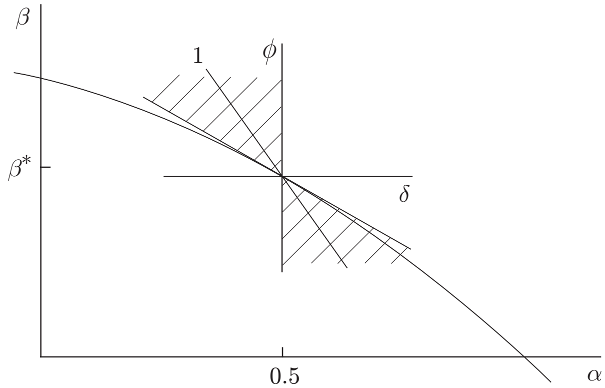

Fig. 4. Domain of existence of solutions at the singular point for $n \neq 1$.

Thus, as it follows from Eqs. (37)–(39), two families of solutions with phase velocities $c_1 \neq 0$ branch off from the first spatial family (31), (32)–(39) along the line (38). One of them satisfies the accepted normalization condition ($c_1 > 0$).

In contrast to the first spatial family, where different values of the amplitudes $|A_0 a_0|$ and $|N_0|$ correspond to different values of δ and ϕ, the family of solutions with $c_1 \neq 0$ bifurcates without deviations from line (38) in this approximation (curve 1 in Fig. 4). In this family, identical values of δ and ϕ correspond to different solutions in which the velocity c_1 increases with increasing spatial harmonic $|A_0 a_0|$. It follows from Eqs. (35) and (38) that the absolute values of the amplitudes of three- and two-dimensional harmonics are $|N_0| = |A_0||a_0|$.

Equation (16) derived in the present work can be used to simulate wave processes in downward-flowing films of nonlinearly-viscous fluids. The analytical results of solving Eq. (16) presented here will be used as the initial approximation in numerical calculations of regimes with wavenumbers lying rather far from the neutral curve (19).

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